Interbank Debt Contagion and Financial Network Solvency

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Abstract

I develop a stochastic model to analyze banks' default status in the network and network stability. I set up the loan creation rate and loan removal rate, coupled with the endogenous bank default rate in the model, to analyze the trajectory of surviving banks in the system. For both short-term and long-term equilibriums, I use a debt exposure Markov matrix to analyze solvency thresholds. I prove that my Markov process is ergodic and has a unique invariant stationary state. My shortest path algorithm demonstrates the efficacy of mitigating risk exposure in the network. Finally, I use Bank for International Settlements data to demonstrate the applicability of my model.

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1. Introduction

Rapidly developing technology enables economic connectivity between financial institutions across geographic locations simultaneously, instantly, and inexpensively in global capital markets. High connectivity and interdependency between banks and financial institutions create a massive, complex financial network regionally and globally. Banks have become more reliant than other sources of global funding in the low interest rate environment, especially for financial institutions in advanced economies. Interbank lending or borrowing, or debt or equity financing, may result in the contagion effect if one or a few banks fail in the financial network system. Lehman Brothers' collapse in the 2008 global financial crisis triggered the height of the financial turmoil. In April 2021, the Archegos collapse cost Morgan Stanley $911 million, Nomura $2 billion, and Credit Suisse $4.7 billion[2] , and the losses are still growing as the ripple effect spreads further. Unceasing and devastating financial crises push researchers to examine the risks generated by an interdependency network. Adverse shocks can come from anywhere in the network system; e.g., they can come from the banking sector internally via interbank lending channels. They can be sparked from the collapse of the asset side or liability side of a bank's balance sheet, or can be caused by failure of a bank's counterparties via direct or indirect interdependency. They can also come from a non-bank financial institutions (NBFIs) externally via debt or equity financing channels.

One of the key deficiencies of measuring the financial risks before the global financial crisis is the computation of risk measures at the individual firm level or at the aggregate level without considering interdependency of debt exposures between banks. For example, a portfolio of AA bonds may be as risky as a portfolio of A bonds if AA bonds are positively dependent. (Gouriéroux et al. 2012, Cifuentes et al. 2005). Collateralized loan obligations (CLOs) are another example. CLOs are securities backed by a pool of loans and are repackaged and sold to different investors. CLOs are structured in tranches with different priorities in terms of the cash flows from the underlying assets. However, investors may have less or asymmetric information for the underlying assets than issuers. In particular, the interdependency of the underlying assets of a CLO may not be taken into account by both issuers and investors. To mitigate the systematic risk, Basel III was implemented in 2009. It requires banks to maintain appropriate leverage ratios and hold certain levels of reserve capital. However, it is difficult to measure a mixed portfolio with different rating scales, and Basel III is still far from sufficient to mitigate all these potential risks.

How can we develop a strategic approach to identify vulnerable banks in the whole network system? How do we quickly identify the first failure bank and the first wave of the cascade failure in a network system? What is the short-term and long-term steady state equilibria for the banking sector? From an individual bank's perspective, how can it choose an optimal path to minimize debt exposure risk? All these questions can shed light for policy makers and banking regulators in mitigating the contagion effect and solvency risk in order to maintain financial stability.

In this paper I develop a novel dynamic stochastic model that provides new insights and can catch interdependent debt exposures in both short-term and long-term market equilibria for

[1] Financial Times, Stephen Morris, "Morgan Stanley reports $911m Archegos losses," April 16, 2021.

the banking sector. This paper's results show that there exists a long-term steady state equilibrium in the network system. The number of surviving banks in the network system is non-linear and non-monotonically associated with network density. The model can identify who fails first, the cascade sequence, and the shortest path of investment in the network.

There are three novel contributions in this paper. First, unlike most traditional approaches in financial network stability literature, I use stochastic models to estimate debt exposure and sovereign risks of the banking system. I build a stochastic model by considering the rate of creating new loans, the rate of removing the existing loans, and endogenous bank default rate, and how the changes of these rates affect banks' balance sheets. The stochastic master equation catches the dynamic movement of banks' solvency paths in the short-run and long-run steady state equilibrium, and the model estimates the total number of surviving banks over time in the network.

The second key contribution of this paper is to employ Markov chain theory to estimate the long-term steady state of the network. Markov chain theory assumes that a bank's next status does not depend on its status in the past, but only on today's status. The Markov chain model provides us the long-run debt exposure of each bank in the network. If we say that Markov chain models forecast the future of debt exposure for us, then the third shortest path algorithm tells us the short-term strategy of each bank. The third main contribution of the paper is to use the shortest path algorithm in the network to estimate the optimal investment paths or minimal debt exposure.

Solvency risk is present everywhere within a network system. Banks can lend or borrow with each other via debt financing channels, or holding other banks' equities via equity finance. Both debt finance and equity finance affect associated banks' balance sheets. Each bank has its own balance sheet's characteristics, connectivity, and asset size position in the network system. A failure of a bank due to an adverse funding shock or credit shock would result in a knock-on effect on other associated banks. In the short run, this shock's initial impact may concentrate among limited banks that link with the defaulted bank. In case these linked banks perish, the previously unaffected banks are brought into the front line of contagion with increased danger of premature liquidation of long-term assets and the associated loss of value. In the long run, failing banks will no longer exist in the network, and the surviving banks' asset or equity share may converge to a long-term steady state equilibrium.

The literature has made significant progress in analyzing and modeling interbank network systems, especially after the 2007–08 global financial crisis. One group of researchers, Erdös and Rényi (1959), Eboli (2007), Song et al. (2014), Eisenberg and Noe (2001), Elsinger et al. (2013), Soramäki et al. (2006), and Bech and Atalay (2010) employ theoretical mathematic models or topology theory to analyze interdependency and connectivity of bank failure, and these models provide a solid foundation for empirical studies. Another school of researchers, Chan-Lau (2010a, 2010b), Minoiu and Reyes (2011), Espinosa and Sole (2014 2014), Furfine (1999), Iyer and Alcalde (2006), Nier et al. (2007), Sheldon and Maurer (1998), Degryse and Nguyen (2004), Wells (2002), Degryse et al. (2009), and Almeida (2015), employ a balance sheet approach to analyze country-level debt holding and exposures; in particular, the risk transmission channels of bank failure across borders.

Another stream of literature on financial risk, Diamond and. Dybvig (1983), Allen and Gale (2000), Elliott et al. (2014), Anand et al. (2012), Freixas et al.(2000), Glasserman and Young (2016), Gofman (2017), Gai and Kapadia (2010), Gai et al. (2011), Aldasoro et al. (2016), Drehmann and Tarashev (2013), Rochet and Tirole (1996), Song et al. (2014), Ghamami et al.(2019), Anderson et al. (2019), Calomiris and Carlson (2017), Nier et al. (2007), Leitner (2005), and Haldane (2009), analyzes financial contagion from a theoretical perspective. They analyze the relationship of network structure and banks' solvency risk, and conclude that the financial network systems exhibit a robust yet fragile tendency; that is, within a certain range, interbank density connectivity serves as shock absorbers and mitigates the amplifying effect of the shock. However, beyond a certain range, interconnections start to serve as a mechanism for propagation of shocks.

This paper is most closely related to empirical studies of contagion: Elliott et al. (2014), Gai and Kapadia (2010), Gouriéroux et al. (2012), Acemoglu et al. (2012, 2015), and Anand et al. (2012). Acemoglu et al. (2012) argue that from a social planner's perspective, moderate shocks corresponding with a perfectly diversified pattern of cross-holdings may be optimal, while very large shocks with perfectly diversified holdings may be the worst scenario. Gai and Kapadia (2010) conclude that large shocks may have devastating consequences, and a shock's impact varies depending on where in the network it hits and on the connectivity of the network. Elliott et al. (2014) further extend Gai and Kapadia (2010) with Acemoglu et al. (2015) by focusing on a moderate shock, which has not been well quantified in previous literature, and argue that intermediate levels of cross-holdings integration[3] and diversification may be problematic. They focus on how a bank's cross-holdings equity debt exposure is held within the banking networks or outside the banking networks (final investors) and what the distribution of debt exposure is within banking networks. Anand et al. (2012) focus on how bad news can lead banks to lose confidence and withdraw their deposits or lending, in turn triggering bank failure in the financial network.

Based on their seminal studies, I develop the network model to analyze the cascades of bank failures in the banking system. The model examines how the change of equity cross-holdings share interacts with bank failure under adverse shocks. I further extend the model by using the Markov chain process to predict the long-term network steady state equilibrium.

I take up this challenge by introducing a Markov chain stochastic process to analyze both short-term and long-term network steady state debt exposure equilibrium and to estimate the minimum path using a shortest path algorithm. These analytical methodologies or approaches may open other new territory in analyzing financial networks.

2. Banking Sector Theoretical Network Model

2.1. Balance Sheet of Banking Sector

[3] According to Elliot et al. (2014), integration refers to the level of exposure of firms to each other—how much of a firm is privately held by final investors, and how much is cross-held by other organizations. Diversification refers to how spread out cross-holdings are: Is a typical firm held by many others or by just a few?

The economy has three dates (t = 0,1,2) and there is no discounting. I assume that matrix $\boldsymbol{V_i}$ represents the market value of total equity, with all elements $a_{ij} \in \{0,1\}$. a_{ij} represents bank *i*'s claim on bank *j*, and the sum of the share of bank *i*'s claims across all its counterparties and its primitive assets equals 1. $\sum_j a_{ij} = 1$.

(Equation 1) $$V_{ij} = \begin{bmatrix} a_{11} & \cdots & a_{1j} \\ \vdots & \ddots & \vdots \\ a_{i1} & \cdots & a_{ij} \end{bmatrix} = (a_{ij}) \in \mathbb{R}^{i\ x\ j}$$

(Equation 2) $$V_i = \sum_{\boldsymbol{k}} \boldsymbol{A_{ik}} \boldsymbol{p_k} + \sum_{\boldsymbol{j}} \boldsymbol{E_{ij}} \boldsymbol{V_j} + \boldsymbol{Q_i}$$

As shown in Equation 2, at time t = 0, equity value V_i of bank *i* equals its own primitive assets A_i plus its claims on its counterparts E_i and liquid asset Q_i; a liquid asset can be cash or other cash equivalent assets such as treasury bills. For simplicity, we assume that bank *i* only has one non-liquid asset *k*. V_j represents assets held by bank *j*.

Where $\boldsymbol{p_k}$ is the market price of bank *i*'s asset *k* at time 0, $\boldsymbol{A_{ik}}$ denotes the share of the value of asset *k* held by bank *i*, and $\boldsymbol{E_{ij}}$ is the fraction of bank *i*'s claim on bank *j*. For any $i, j \in N$, $\boldsymbol{E_{ij}} \geq 0$, where $\boldsymbol{E_{ii}} = 0$ for each bank *i*. If bank *i* has a positive claim on bank *j*, then $\boldsymbol{E_{ij}} > 0$. Let $\boldsymbol{\widehat{E}_{ii}} = \boldsymbol{1} - \sum_{\boldsymbol{j \in N}} \boldsymbol{E_{ji}}$, where $\boldsymbol{\widehat{E}_{ii}}$ represents the fraction of the asset value of bank *i* that is owned by outside shareholders. Following Elliott et al. (2014), I set off-diagonal entries of the matrix $\widehat{E}$ to 0. Bold font stands for a vector or matrix.

Equation 2 can be written in matrix notation at time 0 as

(Equation 3) $$\boldsymbol{V} = \boldsymbol{Ap} + \boldsymbol{EV} + \boldsymbol{Q}$$

Rearrange Equation 3, and we get

(Equation 4) $$\boldsymbol{V} = (\boldsymbol{I} - \boldsymbol{E})^{-\boldsymbol{1}}(\boldsymbol{Ap} + \boldsymbol{Q})$$

Briosch et al. (1989), Fedenia (1994), and Elliott et al. (2014) argue that the "market" value of an organization can be captured by the equity value of this organization that is held by its outside investors. $\widehat{E}_{ii}$ denotes this fraction of remaining shares held by outside investors, and this portion is outside the interbank network system. At time 0, if $\widehat{E}_{ii}$ = 0, then bank *i* has no outside shareholders and bank *i* is a holding company. The market value $v_i = \widehat{E}_{ii} V_i$, $\boldsymbol{\widehat{E}} = \boldsymbol{I} - \boldsymbol{E}$, therefore

(Equation 5) $$\boldsymbol{v} = \boldsymbol{\widehat{E}V} = \boldsymbol{\widehat{E}}(\boldsymbol{I} - \boldsymbol{E})^{-\boldsymbol{1}}(\boldsymbol{Ap} + \mathbf{Q})$$

Let $F = \boldsymbol{\widehat{E}}\,(I - E)^{-1}$ and substitute it into Equation 5, and this yields

(Equation 6) $$\boldsymbol{v} = \boldsymbol{\widehat{E}V} = \boldsymbol{F}(\boldsymbol{Ap} + \boldsymbol{Q})$$

I define *F* as the debt exposure matrix, and define the total share values of bank *i*'s claims on all its counterparts and its own underlying assets as equal to 1. That said, for all $j \in N$, we have $\sum_{i \in N} F_{ij} = 1$. We assume that *F* matrix is always an *N* by *N* square matrix, the number of rows equal to the number of columns.

2.2 Equilibrium, Disequilibrium, and Multiplicity

A bank may suffer unexpected internal or external adverse shocks; when total values of a bank fall below certain critical thresholds, the bank has to go through a liquidation process and these discontinuities of banks in the system can lead to a cascading failure and may also generate multiple equilibria. Gouriéroux et al. (2012) and Elliott et al. (2014) prove that multiple equilibria exist when banks suffer an exogenous adverse shock.

Liquidity risk is one of the most important topics when we analyze banking sector financial stability. Cifuentes et al. (2005) argue that liquid assets are exhausted first before illiquid assets are kicked in when a fire sale occurs. If prices of illiquid assets continue to decline and total assets drop below the obligation payment level or the specific threshold, banks are forced to initiate a bankruptcy or liquidation process.

Bankruptcy cost and asset recovery rates have been studied often in finance literature. Glasserman and Young (2016) argue that when a bank defaults, it would postpone its payment to its creditors as well as for legal and administrative costs. Thus, it can only pay a limited portion of its available assets to its creditors. And the costs may escalate in terms of both the magnitude and the likelihood of default cascades. Rogers and Veraart (2013) extend the Eisenberg-Noe framework to illustrate how bankruptcy cost is associated with bank asset recovery rates. Following the approach of Glasserman and Young (2016) and Elliott et al. (2014), I assume that $\Lambda_i(v,p)$ represents the total payments (Λ) to creditors as a function of the bank's assets v and its nominal obligations (p).

At time 1, when banks start to feel the stress due to adverse shocks, the stressed banks' equity prices may start to fall. Many economists regard a bank's failure cost to be approximately equal to a bank's market value. Liquid assets, especially short-term assets, are the first employed to meet the payment obligation. A fraction of consumers who experience a liquidity shock may tend to withdraw their deposits. A portion of a bank's remaining assets are employed to pay the remaining depositors. As long as a bank's holding of the short-term assets can meet the withdrawals of consumers who withdraw deposits at time 1, the bank can still remain in solvency. That said, if a bank's total assets are at least greater than the total failure cost and payment obligation, then this bank survives.
(Equation 7)

$$V_i = \sum_{\boldsymbol{k}} \boldsymbol{A_{ik}P_k} + \sum_{\boldsymbol{j}} \boldsymbol{E_{ij}V_j} + \boldsymbol{Q_i} - s\boldsymbol{\Lambda}_i(v,p) \geq 0$$

Where Λ_i denotes failure cost and *s* is a binary number: when *s* =1, bank *i*'s value is greater than or equal to the threshold level, otherwise, $s = 0$.
In matrix notation, Equation 7 becomes
(Equation 8-1)

$$\boldsymbol{v} = \widehat{\boldsymbol{E}}(\boldsymbol{I} - \boldsymbol{E})^{-\mathbf{1}}(\boldsymbol{Ap} + \mathbf{Q}) - s\boldsymbol{\Lambda}_i(v,p)$$

A bank's total market value of its assets can be reflected via the prices of assets sold, as in Cifuentes et al. (2005), thus we can employ the following inverse demand function to represent banking assets:

(Equation 8-2) $P(x) = \exp(-\alpha x)$

Where $\boldsymbol{P}$ stands for the price of banking assets and x denotes the aggregate demand for assets. The constant α represents the speed of banking assets' decline. It also implies that the price of banking assets is unity if no assets are forced to be sold and approaches 0 when demand for a bank's assets goes to infinity.

At time 2, if the liquidity shock is large enough and a stressed bank's liquid and short term assets may deplete ($Q_i = 0$), the bank may be forced to use its long-term assets to meet its payment obligations. Investors may then be aware of the bank's liquidity constraint and start to withdraw their deposits, causing prices of long-term illiquid assets to plunge. When the total market values of long asset decline to a certain threshold level such that total assets are less than total failure cost, the bank is in insolvency status. If the bank fails to make the payment for a certain period the bank will default and s =1.

(Equation 9) $\boldsymbol{V} = \ \widehat{\boldsymbol{E}}\ (\boldsymbol{I} - \boldsymbol{E})^{-\mathbf{1}}(\boldsymbol{Ap} + \boldsymbol{Q}) - \ s\Lambda_i(v,p) = \boldsymbol{F}(\boldsymbol{Ap} + \boldsymbol{Q})) - \boldsymbol{s}\Lambda_i(v,p)$

Equation 9 represents an equilibrium set of values for banks based on debt exposure matrix *F*. Denoted $\mathbf{1}_{\boldsymbol{i}}$ the indicator variable that takes the value 1 if a liquidation event occurs at bank *i* and 0 otherwise.

Consistent with other literature (Morris and Shin (2003) and Cifuentes et al. (2005), we assume a bank's decision to lend or borrow is independent across the banking system.

Lemma 1: In any finite connected banking network, given price *p*, if at least one bank has positive equity value, then there is a unique clearing vector v such that

(Equation 10) $\mathbf{1}_{\boldsymbol{i}} = \begin{cases} 1 & \text{if } v = \boldsymbol{F}(\boldsymbol{Ap} + \boldsymbol{Q}) - s\boldsymbol{\Lambda}(v,p) \quad < \mathbf{0} \\ 0 & \text{otherwise} \end{cases}$

(Equation 11) $\boldsymbol{H}(\boldsymbol{v},\boldsymbol{p}) \equiv \boldsymbol{F}(\boldsymbol{Ap} + +\boldsymbol{Q}) - s\boldsymbol{\Lambda}(v,p)$

Equation 10 implies that if $\boldsymbol{F}(\boldsymbol{Ap} + \boldsymbol{Q}) - s\boldsymbol{\Lambda}(v,p) \ > 0$, bank *i* survives. Otherwise bank *i* defaults. Equation 11 shows that there is at least one fixed point of *H* (.) satisfying Tarski's fixed point theorem (Tarski 1955), and at least one clearing vector v. Eisenberg and Noe (2001) have proved that under regularity conditions, there is a unique fixed point of such a function.

Liquidation equilibrium may exist when a bank's failure cost equals its assets as shown in Equation 10. When failure cost is greater than total assets, discontinuity occurs when a bank can no longer operate its business normally. In the banking network system, Gouriéroux et al. (2012) and Elliot et al. (2014) have proved that there exist multiple solutions to the valuation. Gouriéroux et al. (2012) demonstrate four regimes using two banks as an example: no default, joint default, and default of bank 1 only, and default of bank 2 only. Sources of equilibrium multiplicity that satisfy Equation 10 can be caused by joint default or at least one bank's default.

An entry F_{ij} of the debt exposure matrix catches at least two liquidation costs if bank *j* defaults. The first type of cost is bank *i*' s direct claims on the primitive assets that bank *j* directly holds. The second type of cost is bank *j'*s failure cost that bank *i* has to indirectly bear. That said, each bank has two states: default and not default. And the ripple effect of each bank is propagated to the banks that have linkages with it.

There are four steps of the calibration algorithm that we use in this debt exposure network model. First, set up a direct network graph G with N nodes and estimate the number of inflow links and outflow links. Second, construct the debt exposure matrix F according to the equations above. Third, select a bank and set its price of asset value equal to 1 ($p_{ij} = 1$), normalize the matrix using this bank's price, and set threshold asset value $\underline{v}_i = \theta v_i$, where $\theta \in (0,1)$. In this study, we set $\theta = 0.5$. Fourth, select one bank and set its $p_{ij} = 0$ while price value for other banks equals 1, and calculate the best case equilibrium.[4]

2.3 An Example of a Debt Exposure Matrix

Let's use a three-by-three matrix as an example to illustrate how the debt exposure matrix is constructed. Matrix E represents a bank's claim on all its counterparts including itself. And we set $E_{ii} = 0$. $\widehat{E}$ represents a fraction share that is owned by outside shareholders. Matrix F represents the cross-holding of each of three banks on the other banks, where $F = \widehat{E}(I - E)^{-1}$. Debt exposure matrix F demonstrates both the direct holdings and cross-holdings of assets for each bank. The column sum of matrix F equals 1.

$$E = \begin{pmatrix} 0 & 0.66 & 0.21 \\ 0.47 & 0 & 0.63 \\ 0 & 0.07 & 0 \end{pmatrix} \widehat{E} = \begin{pmatrix} 0.53 & 0 & 0 \\ 0 & 0.27 & 0 \\ 0 & 0 & 0.16 \end{pmatrix} \quad F = \begin{pmatrix} 0.79 & 0.56 & 0.52 \\ 0.20 & 0.42 & 0.31 \\ 0.01 & 0.02 & 0.17 \end{pmatrix}$$

Figure 1A. Weighted graph of $W = E + \widehat{E}$

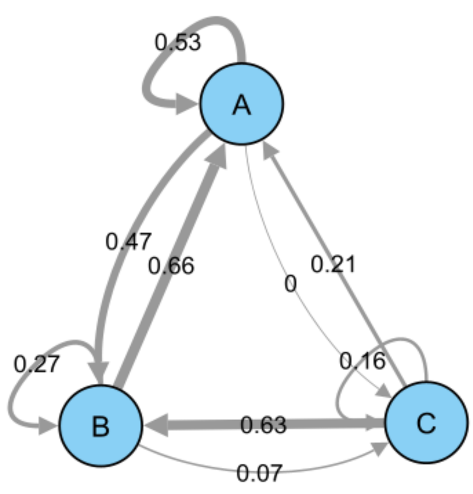
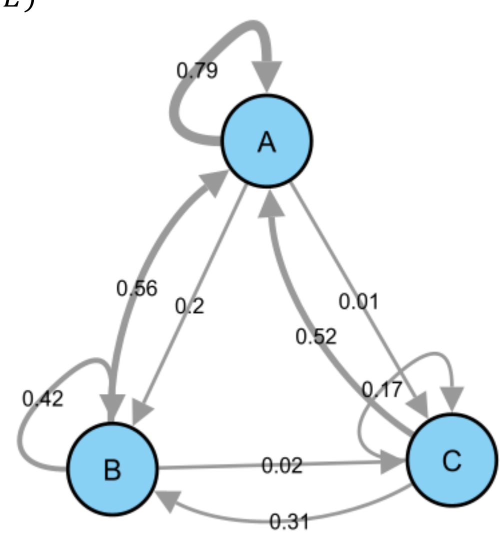


Figure 1B. Weighted graph of $F = \widehat{E}\,(I - E)^{-1}$



Source: Author's configuration

[4] "Best case equilibrium" is defined in Elliott et al. (2014), which is one of the scenarios that illustrates cascades of failure due to positive cross-holdings with the discontinuities in values between organizations. It refers to a case of the minimum possible set of banks/financial institutions that could fail after a shock.

Note: The widths of the edges are proportional to the shares of cross-holdings; the arrows point in the direction of the flow of assets—from the bank that is held and to the holder. Figure 1A demonstrates the shareholders' cross-holdings (E) and outsider's share holdings $(\widehat{E})$, which is represented by self-loops in the charts. Figure 1B shows the dependency matrix, which represents how each bank's equity is distributed among all its counterparts.

The debt exposure matrix demonstrates not only the exposure in interbank lending but also each bank's exposure that is external to the banking system. As shown in Figure 1A and Figure 1B, $F_{ii} > W_{ii}$; that said, the share values in the diagonal of matrix *F* are always greater than the share values in the diagonal of matrix *W*, because debt exposure matrix *F* includes both a bank's direct claims and indirect claims, while matrix *W* only has a bank's direct claims. Another feature that we should notice is that even though Bank C has no direct claims on Bank A (Figure 1A), it does have ultimate debt exposure on Bank A, with 1 percent that contributes from indirect claims (Figure 1B). These two figures show that outside shareholders of Bank A have direct and indirect claims on 52 percent of Bank C's direct asset holdings, though it has only 21 percent of Bank C's shares directly in cross-holdings.

3. Stochastic Process of Banking Network

3.1. Baseline Stochastic Model

In this section, I want to introduce a Markov chain to catch the banking sector's long-run path. A Markov chain is a stochastic process with the property that the future state of the system is dependent only on the present state of the system and conditionally independent of all past states. As shown in Figure 2, after an adverse shock, banks move between the asset side called B space and the liability side denoted L space. At time *t*, let n_1 be a number of banks in state B, while n_2 is the number of banks in state L. The total number of banks in the system is *N*, $N = n_1 + n_2$, which is a constant number.

The threshold for each bank's initial position in the lattice $\mathbb{P}$ (b, l) is the bank's initial assets-to-liabilities ratio. Any adverse shock to any bank in the lattice may affect the bank and the associated banks' balance sheets such that their initial states in the lattice may change. Some banks may survive and some may not. Using Lemma 1 that we discuss in section 2, we can write the threshold for banks' movement in the lattice as follows:
(Equation 12) $\Phi = \frac{b_i+q_i}{l_i}$,
Conditions $\Phi > 1$ or $\Phi < 1$ or $\Phi = 1$ represent three different states of bank *i* in the lattice. If bank *i*'s $\Phi < 1$, this bank falls into Liability (L) state, otherwise, it is in B state, where the bank lies above the 45-degree line. When $\Phi = 1$, it stays at the 45-degree line. Our analysis mainly focuses on movement between $\Phi \geq 1$ and $\Phi < 1$ states.

Banks' transition of states may go only from n to $n + 1$ or from n to $n - 1$ on the lattice. Let's assume that a bank removes a loan or does not renew a loan at a rate φ^- per unit of time, and a bank issues a new loan or renews an existing loan with its counterparties in the system at rate φ^+ per unit time, and $0 \leq \varphi^- \leq 1, 0 \leq \varphi^+ \leq 1$. Please note that issuing a new loan or renewing the existing loan implies adding a link, and removing a loan means cutting a link.

Denote ξ as the negative shock arrival rate per unit time. All loan addition (φ^{+}) and loan reduction (φ^{-}) distributions are Poisson processes and mutually independent stochastic variables, and state transitions between two states occur through exactly one bank moving to each direction at a time. The overall number of added and removed loans in the network system is $\varphi = \varphi^{-} + \varphi^{+}$.

Consider a large class of finite state space, a continuous-time Markov chain $\{X(t): t \in [0, N]\}$ whose state spaces are finite $[0,1,2, .. N]$. Assume that its transition probabilities $P_{ij}(t)$ are stationary; that is:
(Equation 13) $P_{ij}(t) = P(X(t+s) = j | t(s) = i) \ \ \forall \, \Delta t \geq 0$

For the boundary solution, we assume that if there are no banks in the network system, then the removing rate φ^{-}= 0. $X(t)$ is allowed to increase or decrease. If at time *t* the process is in state asset (B) side, after a random sojourn time it may move to liability side (L). The process is irreducible if and only if all the $\varphi_n^{+} \geq 0$ and all the $\varphi_n^{-} \geq 1$.

We want to find out the probability $P(n)$ of having n banks in state B, L at time *t*. That is, the master equation that provides a trajectory of the total probability density function of total bank movement over time. It can be obtained by considering all transitions that can take place during the time interval between t and $t + \Delta t$. There are n banks in state B and $N - n$ banks in state L.

Let's list the three possibilities that a bank may stay and move between asset state (B) and liability state (L). First, there are $n + 1$ banks at state B at time *t*, and one of them defaults due to adverse shock and moves to state L during the time interval $(t, t + \Delta t)$. Any of these $n + 1$ banks in state B has a probability $P_{n+1}(t + \Delta t) = (\varphi^{-})\Delta t + O(\Delta t)^2$. The total possibility that any bank may move is $(n + 1)\ (\varphi^{-})\ \Delta t$. We assume there are no more than two banks jumping from state B to state L. The notation $O(\Delta t)$ represents some function that is much smaller than Δt for small Δt.

For the second possibility, there are $n - 1$ banks at state B (asset side) at time t, and one of the $N - n + 1$ banks at state L (liability side) moves to state B. Let $j \in N.$ A bank may reduce its debt or increase its lending such that it improves its balance sheet resilience due to external shock. This state change can be expressed as: $P_{n-1}(t + \Delta t) = \{[N - (n + 1)]\varphi^{+}\Delta t\}^{N-n+1} = (N - n + 1)\varphi^{+}\Delta t + O(\Delta t)^2$, where $O(\Delta t)^2$ represents a bank moving from state B (asset side) to state L and from state L to state B. We can estimate these jumps $O(\Delta t)^2 = \varphi^{-}dt * \varphi^{+}dt$. We exclude the higher order terms such as more than two banks moving from one state to the other state. Similarly, the third possibility occurs as follows: there are n banks at state B at time t and no bank moves between the two states. It can be shown as $1 - P_n(t + \Delta t) = 1 - [n(\varphi^{-} + \varphi^{+})\Delta t + O(\Delta t)^2]$.

Since no other possibilities take place, we sum them up together as the master equation:
(Equation 14A) $P_{n,n+j}(t + \Delta t) = Prob\{X(t + \Delta t) - X(t) = j |\ X(t) = n\}$

$$= \underbrace{P_{n+1}(t)(n+1)\varphi^{-}\Delta t + O(\Delta t)^2}_{j=n+1} + \underbrace{P_{N-n+1}(t)(N-n+1)\varphi^{+}\Delta t \ + O(\Delta t)^2}_{j=N-n+1}$$

$$+\underbrace{P_n(t)\{1-[n(\varphi^-+\varphi^+)\Delta t]\}+O(\Delta t)^2}_{j=n}$$

Rearranging this equation,
(Equation 14B)

$$P_n(t+\Delta t)-P_n(t)=P_{n+1}(t)(n+1)(\varphi^-)\Delta t+P_{N-n+1}(t)(N-n+1)\varphi^+\Delta t$$
$$-P_n(t)n(\varphi^-+\varphi^+)\Delta t+\ 3O(\Delta t)^2$$

Dividing both sides by Δt, taking the differential of the equation, rearranging this equation and taking the limit $\Delta t \to 0$, the master equation without considering external shock on the balance sheet can be written as follows:
(Equation 15)

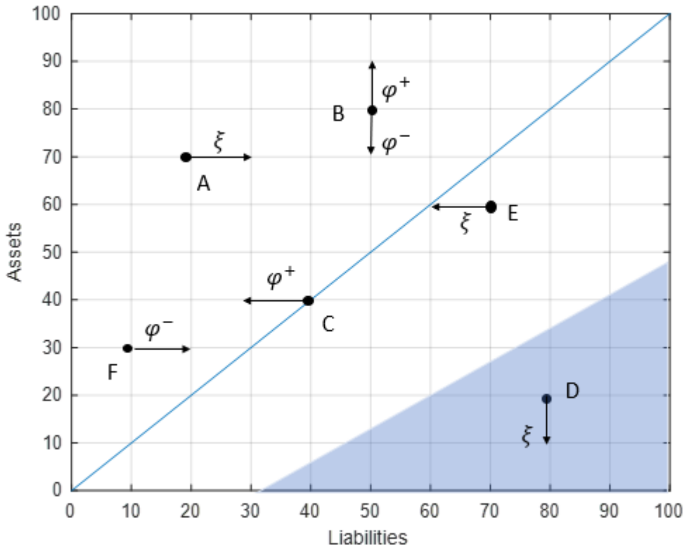
$$\frac{\partial P_n(t)}{\partial t}=\varphi^-(n+1)P_{n+1}(t)+\varphi^+(N-n+1)P_{n-1}(t)-n[\varphi^-+\varphi^+]P_n(t)$$

$$\frac{\partial P_0(t)}{\partial t}=\varphi^-P_1(t)$$

This master equation describes the marginal distribution function of the fraction of surviving banks in the banking system. Appendix A provides the values of initial moment, mean, and variance under different adding and removing loan rates.

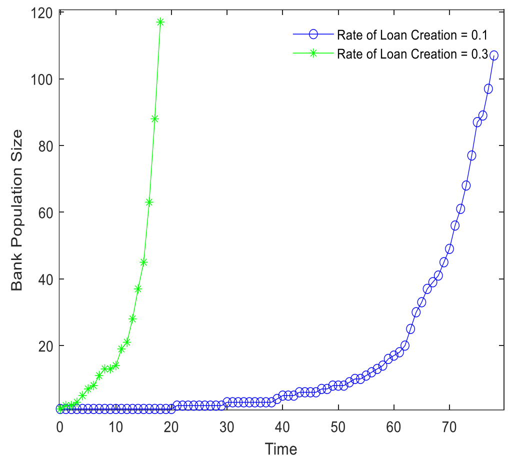
Using master equation 15, we can conduct the dynamic stochastic process analysis. The three simulation charts demonstrate the surviving banks with different creation rate or loan removal rate under different scenarios based on Markov Chain death-birth process. Figure 3A is the pure bank creation model. We start with five banks, each with probability φ^+ of splitting in per unit time. We continue until bank size reaches a target number. Assuming banks add new loans or increase links per unit time with its counterparties, without removing loans, the total number of surviving banks increases dramatically. Figure 3B demonstrates the banks with only rate of loan removal. Starting with 100 banks, each having probability φ^- of removal loan in each time step, and we continue until all loans have been removed. The higher the removal rate, the fewer the surviving banks at a given time. In the bank creation and removal model (Figure 3C), both creation rate and removal rate are endogenously determined by the master equation. We set the creation rate equal to 1 and the removal rate at 0.4 and 0.5 respectively, and initial bank population is 5. Figure 3C shows that lower removal rate has more surviving banks at a given time than those with higher removal rate.

Figure 2. Dynamic Adding and Removing Loan Rates Movement on Balance Sheet



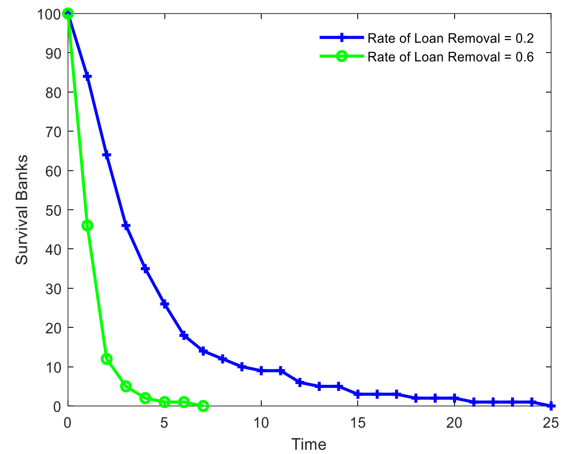
Note: The diagram shows how loan creation rate (φ^{+}), loan removal rate (φ^{-}), and stochastic shock ξ caused by endogenous bank default rate affect a bank's balance sheet given its initial balance sheet status in the network. The 45-degree line shows the break-even line between the asset side (left of the 45-degree line) and liability side (right of the 45-degree line). Bank A's, Bank B's, and Bank F's assets-to-liabilities ratios are greater than 1 and stay at the left side of the break-even line. Bank C's ratio is equal to 1 and lies on the break-even line. Bank D's and Bank E's stay to the right of the break-even line. A stochastic shock ξ may propagate through the network. Coupled with loan creation rate or loan removal rate given a bank's initial balance sheet status, the shock is unevenly distributed cross the network. For example, Bank D is more vulnerable to an adverse shock than other banks, and Bank E benefits by this adverse shock. The shadow area corresponds to Equation 13 where the liquidation process takes place.

Figure 3A. Surviving Banks in Pure Bank with Loan Creation Model



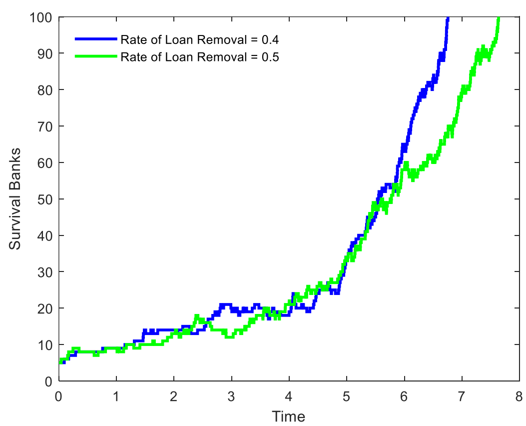
Source: Author's loan creation model estimation
Note: Bank decay rate = 0, initial bank population = 1.

Figure 3B. Surviving Banks in Pure Bank with Loan Removal Model



Source: Author's loan removal model estimation
Note: Bank loan creation rate = 1, initial bank population = 100.

Figure 3C. Surviving Banks in Dynamic Bank with Loan Creation and Removal Model



Source: Author's simulation output
Note: creation rate = 1, initial bank population = 5.

3.2 Endogenous Rate of Bank Failure

Based on the evolution of the interbank network described by master equation 15, we can further derive the master equation under adverse shock using the same approach as Gardiner (2009) and Anand et al. (2012). We assume the endogenous bank default rate μ, and the initial default rate from the asset side and the liability side is $\delta_{b,0}$ and $\delta_{l,0}$ respectively. The probability distribution function includes the joint probability of balance sheet states (l_i^t, b_i^t), and the marginal distribution function $P(l_i^t, b_i^t)$ of the evolution of the fraction of banks with $l_i^t = l$ and $b_i^t = b$ can be derived as follows:

(Equation 16)

$$\frac{\partial P_n(t)}{\partial t} = \mu\delta_{b,0}\delta_{l,0} + \varphi^-(n+1)P_{n+1}^t(l-1,b) + \varphi^-(n+1)P_{n+1}^t(l,b-1)$$
$$+(\mu_b + \varphi^+)(N-n+1)P_{N-n+1}^t(l+1,b)$$
$$+(\mu_l + \varphi^+)(N-n+1)P_{N-n+1}^t(l,b+1)$$
$$-\left[\xi\Theta\left(l^\xi - b^\xi - q^0\right) + 2\varphi^+ + (\mu_b + \varphi^-)l + (\mu_l + \varphi^-)b\right]P_n^t(l,b)$$

Where Θ refers to the Heaviside function, $\Theta(x) = 1$ if and only if $x \geq 0$ and otherwise $\Theta(x) = 0$. Appendix A shows the values of the initial position, mean, and variances from the master equation.

Figure 2 demonstrates the dynamic movement of banks' position according to master equation 16 with adverse shock or liquidity shock. This master equation describes the general trajectory path without external shocks under the Poisson processes.

Suppose there is an external shock that results in banks' balance sheets changing so that banks jump between the asset and liability sides. Denote μ as the endogenous bank default rate; when $\mu = 0$, the probability of default may be low, and when $\mu > 0$, bank default occurs. As we define in the previous section, φ^- =1 refers to per unit rate of a bank moving right or up one unit in Figure 2, φ^+represents the rate of a bank moving to the left or below on the grid. Adding endogenous default rate μ, total removal rate is $\mu + \varphi^-$. Assume that the rate ξ is the critical turning point moment when a bank's liability is greater than its assets, that is

(Equation 16) $\left(b_i^{\xi_t} + q_i^{\xi_t}\right) < l_i^{\xi_t}$

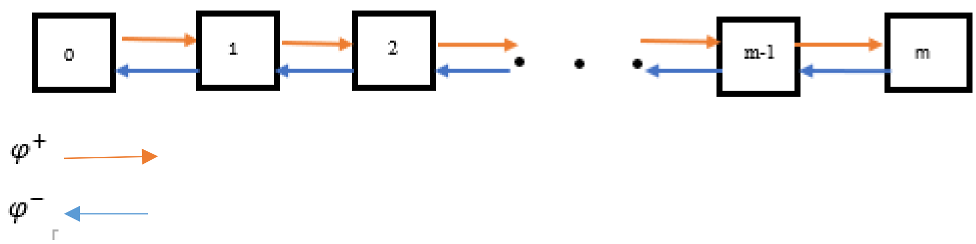
When a bank defaults, ξ_i represents a moment that a bank reaches the threshold point.
Twin stochastic processes ((b^t, l^t) are Poisson distribution variables with mean $1/(\mu + \varphi^-)$. The error function for the Gaussian distribution is erf (x) = $\left(\frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\right)$.

The derivation of the endogenous rate of bank default can be found in Appendix B. There are several key findings from the simulation results. First, density ε is non-linearly negatively associated with threshold Φ. Second, when φ^- is small and threshold Φ is in an intermediate range, there exists two different stationary states (dense vs. sparse network solution). Third, if φ^- is a larger value, it will gradually decline from a dense network to a sparse one when threshold Φ increases. When threshold Φ and ε are small, the magnitude of the shock in the network is limited due to sparse density, and banks have relatively sufficient buffer to defend against the external shock, thus bank failures are rare. These results are consistent with Gai and Kapadia (2010), Allen and Gale (2000), Anand et al. (2012), and Elliott et al. (2014) regarding the relationship between the probability of a bank's failure, its contagion effect, and its position in the network. And the results further quantitatively and qualitatively validate the argument of financial systems exhibiting a robust-yet-fragile tendency.

3.3 Steady State Solution of Banking System

For the general loan addition and removal process, we assume that the limits $\lim_{n\to\infty} P_{ij}(t) = \pi_j \geq 0$ exist and are independent of the initial state i. π_j is called a stationary probability distribution.
Denote $X_i = \varphi P_{ij}$ where X_i represents a state in a set of states.

Figure 4. Dynamic Loan Creation and Loan Removal Rates in the Steady State



Note: Forward arrow moves with speed φ^+ and backward arrow moves with speed φ^-
Source: Author's configuration

Proposition 1. In the long-run steady state, (i) if the rate of the number of new loans added equals the rate of the number of loans removed ($\varphi^+ = \varphi^-$) or $\rho = \frac{\varphi^+}{\varphi^-} = 1$, then the probability distribution at each state is constant, $\pi_0 = \pi_n$, and $\pi_0 = \frac{1}{1+m}$; as the number of states increase, $\pi_0$will decrease.
(ii) If $\rho < 1$, $\pi_0 = 1 - \rho$ and $\pi_n = \pi_0\rho^n = (1-\rho)\rho^n$ and $E(X_n) = \frac{\rho}{1-\rho}$
The proof can be found in Appendix C.

From the network stability perspective, the rate of adding new loans and removing the existing loans can be viewed as updating the network density. Let $\rho = \frac{\varphi^+}{\varphi^-}$ and ρ represents the ratio of the density in the network. ρ is moving between 0 and 1, $\rho \in (0,1)$. When the rate of adding new loans equals the rate of removing existing loans in the network, the whole network system is in the steady state. $\rho < 1$ means the rate of removing an existing loan per unit time (φ^-) is greater than the rate of adding a new loan (φ^+). If the average value of ρ in the whole network system is less than 1, it implies more loans/links are cut than new loans/links are added in the system. The density of the network decreases and moves toward the left side in Figure 4. From the aggregate banking sector perspective, this trend direction may also mean that the liability side is shrinking faster than the asset side. All of these results will be opposite if $\rho > 1$.

4. Stochastic Process of Debt Exposure Matrix

In section 2.3, we derived the debt exposure matrix, which shows that each bank's market value ultimately relies on the assets held by its counterparties. To analyze the stability or long-term equilibrium of the network, we introduce the stochastic process-Markov chain, which is a particular type of stochastic process where only the present value of a variable is relevant for predicting the future. The history of the variable and the status that the present has emerged from the past are irrelevant.

Let's assume a set of states, $X = \{ X_1, X_2, X_3 \dots, X_n\}$, and the Markov chain process starts in one of these states and moves successively from one state to another. A move from one state to another is called a step. Transition probabilities P_{ij} represent the chain moves from state X_i to X_j, and these probabilities do not depend upon which states the chain was in before the current state. The process can remain in the state it is in, and this occurs with probability P_{ii}. Usually we can specify a particular state as a starting state from the set of state X.

The transition matrix is composed of transition probabilities P_{ij}. The debt exposure matrix *F* that we discuss in section 2.3 can be called a transition matrix. The entry of P_{ij} here represents the percentage share of one bank's debt exposure across its counterparties within and outside the banking system. In order to be consistent with Markov chain theory, we transform the matrix such that each row sum, rather than column sum, is equal to 1. This transformation of the matrix does not affect any changes of the output. We can interpret the matrix *F* as follows: For the first row, 79 percent of Bank A's equity is owned by outside shareholders, and 20 percent and 1 percent of Bank A's equity is owned by Bank B and Bank C, respectively. We assume that external shocks S (S_1, S_2, S_3) are random numbers. Note that in this study, we assume that the holders of an equity are the owners of an equity.

Proposition 2. For a finite bank population size *N*, the interbank payment between banking network Markov processes is ergodic and possesses a unique invariant measure.

To prove this proposition, we show that any state $\hat{F}$ can be reached from any state $\ddot{F}$ via a sequence of elementary processes of link addition or deduction. The new state evolves as link reduction or addition. This process exists via a positive probability for both $\hat{F}$ and $\ddot{F}$, and it has a unique invariant measure. The proof of proposition 1 can be found in Appendix D.

According to stochastic process theory, a discrete-time random process $(X_n)_{n\geq 0}$ with initial distribution Γ and transition matrix P is a Markov chain if the following two conditions are met:
(1) X_0 has distribution Γ_0, $P(x_0 = i_0) = \Gamma_0$;
(2) $P(x_{n+1} = i_{n+1} | x_0 = i_0, \dots, x_n = i_n) = P_{i_n i_{n+1}}, i \epsilon I$. where I is called a state-space and i is called a state. This equation means that each event occurs independently. A Markov chain should have the following property:

$$P(\, x_0 = \, i_0, \, x_i = \, i_i, \dots, \, x_n = \, i_n \,) = \Gamma_{i_0} P_{i_0 i_1} P_{i_1 i_2} \dots P_{i_{N-1} i_N}$$

Basically, this property implies that a Markov chain has no memory. Other properties of a Markov chain in the steady state include: First, there is a unique stationary matrix *H* that can be found by solving the equation *H*P* = *H*; second, given any initial state matrix H_0, the state matrix H_k approaches the stationary matrix *H*; and third, the matrix P^k approaches a limiting matrix $\bar{P}$ and is equal to the stationary matrix *H*. Using the properties of the Markov chain, we can find the stationary matrix *H*.

The dependency matrix *F* satisfies all the Markov chain properties above and can be called a regular Markov chain. The matrix *F* is the transition matrix at the initial state, and a one-time external stochastic shock *s* ($s \in N$) may change the distribution of the current state of the banking system in a sequence step. To be consistent with the Markov chain definition, we transform dependency matrix *F* (which is matrix *P* in the first requirement above) so that the row sum equals 1.

$$\boldsymbol{F} = \begin{pmatrix} 0.79 & 0.56 & 0.52 \\ 0.20 & 0.42 & 0.31 \\ 0.01 & 0.02 & 0.17 \end{pmatrix} \; \boldsymbol{F}' = \begin{bmatrix} 0.79 & 0.20 & 0.01 \\ 0.56 & 0.42 & 0.02 \\ 0.52 & 0.31 & 0.17 \end{bmatrix}$$

Solving the equation *H*F* = *H*, we find the long-term equilibrium or steady state vector. Computing successive powers of *F* results in

$$\lim_{n \to \infty} (P)^n = \begin{pmatrix} 0.7265 & 0.2585 & 0.0150 \\ 0.7265 & 0.2585 & 0.0150 \\ 0.7265 & 0.2585 & 0.0150 \end{pmatrix}$$

where the rows are identical and equal to
H = [0.7265, 0.2585, 0.0150].

This implies that after an external shock given the initial state (matrix F), in the long-run steady state, Bank A’s, Bank B’s, and Bank C’s equity share in the whole banking network will be 73 percent, 25 percent, and 2 percent, respectively. Figure 5 shows that it takes a total of nine steps from the initial transition state to reach this final steady state ($n = 9$).

Figure 5. Evolution of Steady State Equilibrium

Figure 5.1. Heat Map -State 0

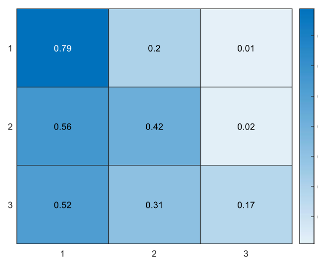


Figure 5.2. Heat Map -State 1

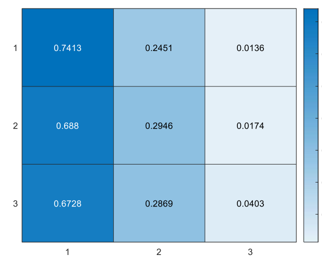


Figure 5.3. Heat Map -State 2

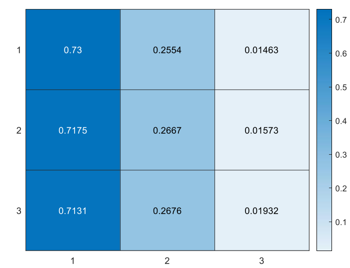


Figure 5.4. Heat Map -State 3

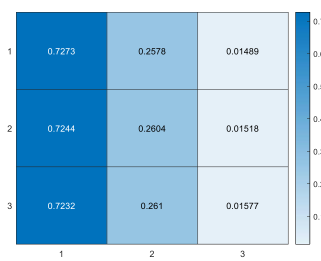


Figure 5.5. Heat Map -State 4

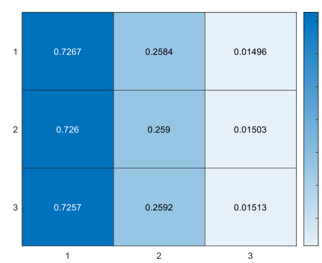


Figure 5.6. Heat Map -State 5

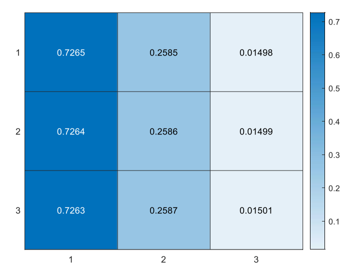


Figure 5.7. Heat Map -State 6

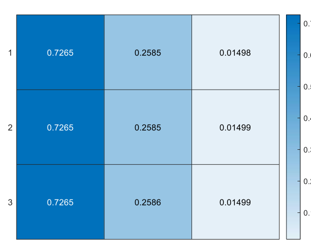


Figure 5.8. Heat Map -State 7

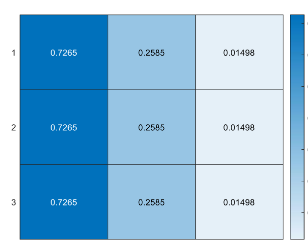
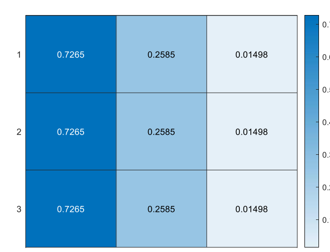


Figure 5.9. Heat Map -State 8



Source: Network model outputs

4.1 Banking Sector Stochastic Network Model Scenarios and Outputs

A bank’s failure can be triggered by each component or multiple components of both the asset side and/or liability side simultaneously due to endogenous and exogenous adverse shocks. Endogenous shocks can be a sudden decline in deposits or increases in non-performing loans, rising interest payments, mismatch of durations, or industry-level specific shocks. Exogenous negative shocks can be a sudden economic recession or turmoil, pandemic, political instability, a natural disaster, or unexpected specific sector business cycle downturn. Other macroeconomic shocks can be an escalation of a trade war, a huge decline of foreign exchange rate, sudden shrinking of foreign exchange reserves, or a sudden interest rate hike. Our scenarios consider both shocks from asset valuation and magnitude, and the model simulations are based on the algorithm in section 2.

I use quarterly cross-holdings data from the Bank for International Settlements (BIS), which shows one country's aggregate banking sector's total foreign claims on another country's banking sector, and it measures the value of immediate borrowers rather than final borrowers. Immediate borrower refers to a direct borrower when a bank is from a country different from that of a final borrower. I choose two sets of country data from BIS. The first set of data includes 14 European countries, and the second set of data includes 24 advanced countries and emerging market economies. From the BIS original data, we obtain the matrix $A(\boldsymbol{i},\boldsymbol{j})$, which shows country i 's claims on country j. We transform the matrix to country *i*'s debt to country *j*, so that the row sum equals 1 rather than column sum equaling 1.

To analyze the stability of the financial network, we first identify who fails in a cascade under different scenarios using the algorithm that we derive in section 2. Then I estimate the long-term equilibrium for two sets of global banking networks. I assume in the long-term equilibrium level, each country's total claims on all its counterparties should be balanced by its total debt held by all its counterparts, and the sum of a country's total investment across all its counterparties equals 1. Thus $A = \boldsymbol{I}$. Total gross domestic product (GDP) can be viewed as its initial value of primitive assets for each country. I normalize each country's GDP by setting one country's GDP in 2019 to 1 and obtain a vector $\boldsymbol{p}$ such that $\boldsymbol{v} = \boldsymbol{FAP} = \boldsymbol{FP}$.

For the 14 European countries network model scenario, since the Eurozone crisis started in 2009 and ended around 2014, I use BIS data from 2014 and 2019 here. 2019 data is treated as the baseline. I assume a country may fail if its total value falls below one half of its initial baseline value. According to the studies by Reinhart and Rogoff (2011) and Elliott et al. (2014), I also set up the foreign-to-domestic holdings ratio by one third so that it is in line with empirical literature.

In the first scenario, we consider the dynamic changes in the ratio of cross-holdings and threshold simultaneously. Cross-holdings ratio allows us to measure a country's foreign investment and foreign debt condition change, and threshold ratio provides us the status of a country's nominal GDP and its debt. By considering both cross-holdings ratio and threshold simultaneously, we may have a better picture of a country's sovereign risk from a macroeconomic perspective.

Figure 6A shows the 14 European countries' network structure based on the debt exposure matrix. I find out only four pairs of edges are not connected among the total 196 edges in this chart.[5] The debt exposure ratios in France and Germany are relatively higher than other countries. The debt exposure matrix shows that the diagonals referring to each country's primitive assets are all greater than 0.5. Table 1A shows the cascade sequence in different waves. Greece is the first to fail due to its higher debt ratio and low primitive assets. Sweden also fails.

[5] None of Belgium, Greece, Greece, and Sweden's debt is held by the Netherlands, Finland, Sweden, and Finland, respectively.

The U.K. is also vulnerable because more countries invest in the U.K. than the U.K. invests in its counterparts. That said, the U.K. has relatively higher liabilities than assets.

For each round of simulation, the cascade sequence may change, but it will eventually converge to the final invariant outcome; in this case it is the state that cross-holding share is 0.5 and θ is 0.95. It is possible that the final equilibrium outcome shows that no country fails, or all countries fail, or some countries fail. In this scenario, there are in total 13 countries failing in three waves. The only country that does not fail is Ireland, which is very surprising.

As shown in Table 1B, in terms of 2019 GDP size, Ireland ranks 10th among these 14 countries, and Ireland's debt exposure is the highest in terms of GDP in 2020Q2. However, its 2019 GDP was 50 percent higher than its 2014 GDP, which is the highest GDP growth rate among the 14 countries. The high GDP growth rate of Ireland offsets its negative debt size compared with its peers, thus Ireland passes the stress test. This scenario tells us that only using debt-to-GDP ratio as a risk measure is not a comprehensive method. This model considers both the price effect and income effect multilaterally over time, which provides more accurate analysis of a country's sovereign risk in the network.

For the second scenario, I run the same simulation as the 14 countries case above using BIS bilateral data of 24 countries. Table 2 demonstrates a subset of simulation results in which the cross-holdings share is 10 percent and 50 percent, respectively. In these two scenarios, Brazil, Greece, and Turkey are the countries that fail the test ultimately. Other countries such as Italy, Sweden, France, Spain, and the U.K. fail in some scenarios, as these countries are vulnerable to adverse shocks due to their high debt ratio. The rest of the other countries are relatively resilient.

Table 3 demonstrates the share of long-term steady state equilibrium for both scenario 1 and scenario 2. The value shares are estimated using the Markov chain algorithm that I discuss in section 2. For example, France has the highest value share in both scenarios, implying that France has the highest equity share dependence ratio with its counterparties compared with its peers in these two groups of countries.

One of the caveats of the model is that the model results in sensitivity to the input data since it measures the relative balance sheet change among peers. For example, we use two years' GDP data and debt exposure data here in the model; if we change the data, the results may slightly change depending on the relative price effect and magnitude effect horizontally or relatively among peers. To improve the accuracy of the model result, we can input the average values or moving average values in the model so that the results reflect the average changes.

Figure 6A. Network of Debt Exposure Matrix of 14 Countries

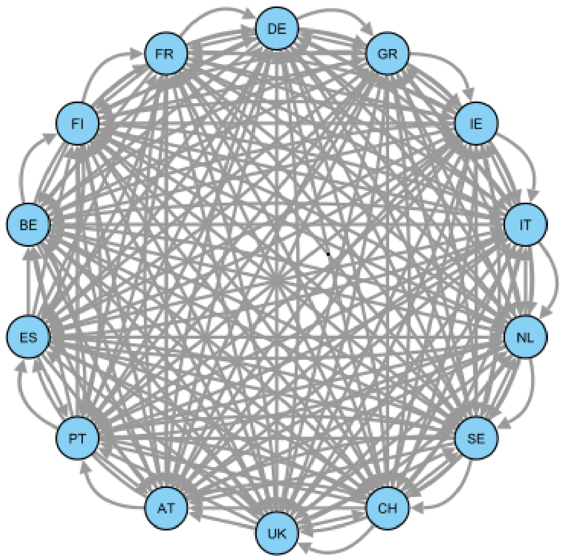
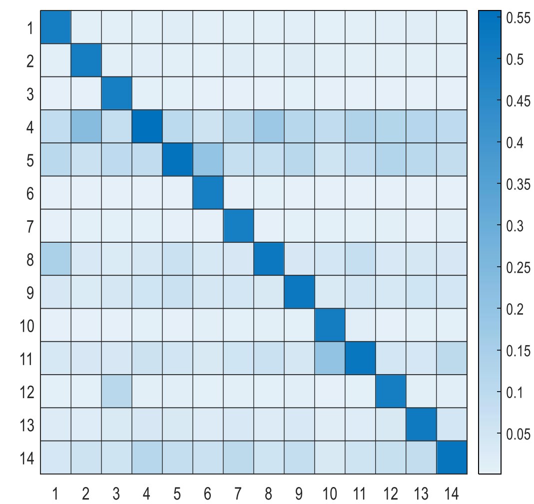


Figure 6B. Heatmap of Dependence Matrix of 14 European Countries



Note: The numbers in Figure 6B stand for the following countries: 1. Austria (AT); 2. Belgium (BE); 3. Finland (FI); 4. France (FR); 5. Germany (DE); 6. Greece (GR), 7. Ireland (IE); 8. Italy (IT); 9. Netherlands (NL); 10. Portugal (PT); 11. Spain (ES); 12. Sweden (SE); 13. Switzerland (CH); 14. United Kingdom (UK)

5. Shortest Path in the Banking Network

We have discussed the contagion effect in financial networks: One bank's failure may lead to other banks' exposure to higher solvency risk. Instead of finding which bank fails first in a network, an investor can construct a debt exposure matrix to find out the minimum cost or highest cost for each pair in the whole network so long as we have bilateral data. I use Dijkstra's Algorithm in analyzing how an investor can find a path with the lowest borrowing cost of a specific portfolio in a network. For example, an investor knows the total investment volume among four countries and also the price of the loan investment, supposing the price is nominal exchange rate. We can construct a four-by-four matrix to estimate the lowest cost of each pair in the network.

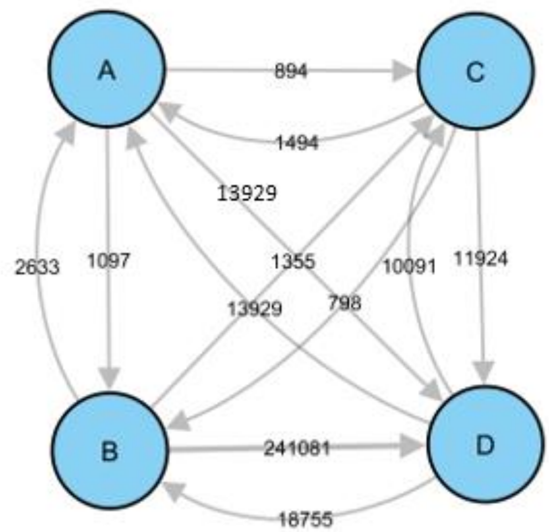
For simplicity, I use randomly generated numbers to represent the debt exposure among these four countries. Figure 7 demonstrates four countries' consolidated foreign claims of banks from one country on debt obligation of another country. Assume that an investor needs to consider the lowest cost to borrow loans from one country to another country. An edge stands for debt or credit. The arrows point to the creditor of assets from the debtor. Suppose that the debt value represents the total cost for one country to another one. Each country can borrow from each other and there is no other transaction cost and no capital control between each pair of these four countries.

Using the shortest path algorithm, what is the lowest cost from country A to country F? Table 4 shows the benchmark cost from each country to another under both the benchmark scenario and the shortest path scenario. The benchmark scenario refers to the original cost between two countries. As we can see, only country F has no change in terms of costs under both scenarios.

The direct cost from country A to country D is \$13,969 million; however, if an investor starts from country A to country C, it would cost her only \$894 million, then from country C to country D, which is another \$11,924 million, thus her total cost or shortest path from country A to country D is \$12,818 million, which is lower than direct cost (\$13,969 million). The same thing holds if an investor makes an investment directly from country B to country D, which would cost her \$241,081 million. However, it costs an investor only \$13,279 million if she takes the shortest path, which goes through country C first, then to country D.

Overall, from a sovereign wealth management perspective, how to mitigate various risks from different channels via choosing the minimum cost path in a global capital market is very important. This is another way of mitigating or identifying the potential highest risk in the network.

Figure 7. Debt Network with Shortest Path (Mil.US$)



Source: Author's configuration
Note: Node size represents a bank's total assets and an edge represents the debt or transaction cost between two nodes.

6. Concluding Remarks

Though there is rich literature related to contagion and solvency risk in financial networks, very little has been studied about both the short-term equilibrium and long-term steady state equilibrium of the banking sector networks. This paper provides a couple of novel methodologies in analyzing banking sector solvency risk and network stability from the perspective of network structure and individual bank solvency. I employ a stochastic model to analyze the trajectory path of surviving banks in the network system. The loan creation rate and loan removal rate, coupled with the endogenous bank default rate, affect both individual banks' solvency and the stability of the network. Consistent with the findings of other literature, I find that the network exhibits a robust yet fragile tendency phenomenon. More importantly, this stochastic model can quantify the relationship between the solvency ratio and network density of the whole network, and can indicate the direction of aggregate balance sheet movement, which provides or serves as early warning signals of network stability.

I derive a debt exposure matrix based on existing literature by incorporating time frame, and extend the model by considering a Markov chain process to analyze the stability of the network in long-term steady state equilibrium. The long-term steady state equilibrium of the debt exposure matrix enables us to forecast potential solvency risk given the network structure, and the debt exposure matrix provides comprehensive information that not only includes individual bank-level balance sheet information, but also the debt exposure relationship between each creditor and debtor in the whole network.

More importantly, I estimate the short-term dynamic equilibrium by identifying the bank that fails first and the cascade sequence of bank failure waves. Using a bilateral debt exposure matrix, we can use the shortest path algorithm to analyze optimal allocation of funds within a network. We can estimate the "debt distance"[6] of each bank with respect to this first failed bank as well, and estimate the solvency risk of each bank in a network.

Further studies can be explored from many aspects along this approach of network analysis. First, since the first bank that fails can be identified through the network, we can also further analyze and forecast the probabilities of the magnitude of the waves and the length of the waves to each node in the network. Second, the question regarding whether bank-run crises are more frequently caused by the interbank lending systems or non-bank financial sectors has not been well studied. Knowing the transmission channel between bank and non-bank financial sectors will allow us to have better ideas in the analysis of contagion risk between sectors. Third, more Markov chain processes such as hitting time, recurrence, and transience can be used to analyze the network stability. Lots of fields remain as uncharted territory for us to explore. As more bank-level data is available now than a decade ago, it is possible for us to digitalize banking sector networks in the near future by using creative methodologies and cutting-edge technologies.

[6] The number of connected and contagion banks between the root/trigger bank and the target bank can be referred to as debt distance. For example, supposing Bank A is a target bank, and there are eight banks that are in the shortest path from root bank to Bank A, then debt distance is eight.

Table 1A. Output of 14 European Country Network Model

Variable name	Wave 1	Wave 2	Wave 3	Wave 4
gamma = 0.10; theta = 0.90	Greece			
gamma = 0.10; theta = 0.95	Greece, Italy, Sweden, U.K.	France	Switzerland	
gamma = 0.15; theta = 0.90	Greece			
gamma = 0.15; theta = 0.95	Greece, Italy, Sweden, U.K.	France, Spain, Switzerland	Germany, Netherlands	Austria, Finland
gamma = 0.20; theta = 0.90	Greece			
gamma = 0.20; theta =0.95	Greece, Italy, Sweden, U.K.	France, Spain, Switzerland	Germany, Netherlands	Austria, Belgium, Finland
gamma = 0.25; theta = 0.90	Greece			
gamma = 0.25; theta = 0.95	Greece, Italy, Sweden, U.K.	France, Spain, Switzerland	Finland, Germany, Netherlands	Austria, Belgium, Portugal
gamma = 0.30; theta = 0.90	Greece			
gamma = 0.30; theta = 0.95	Greece, Italy, Sweden, U.K.	France, Germany, Netherlands, Spain, Switzerland	Austria, Belgium, Finland, Portugal	
gamma = 0.35; theta = 0.90	Greece			
gamma = 0.35; theta = 0.95	Greece, Italy, Sweden, U.K.	France, Germany, Netherlands, Spain, Switzerland	Austria, Belgium, Finland, Portugal	
gamma = 0.40; theta = 0.95	Greece, Italy, Sweden, U.K.	France, Germany, Netherlands, Spain, Switzerland	Austria, Belgium, Finland, Portugal	
gamma = 0.45; theta = 0.95	Greece, Italy, Sweden, U.K.	France, Germany, Netherlands, Spain, Switzerland	Austria, Belgium, Finland, Portugal	
gamma = 0.50; theta = 0.95	Greece, Sweden, U.K.	France, Germany, Italy, Netherlands, Spain, Switzerland	Austria, Belgium, Finland, Portugal	

Note: gamma stands for cross-holdings ratio; theta stands for threshold
Source: Network model estimation

Table 1B. Nominal GDP and Debt Exposures (Mil. US$)

Country	GDP Ratio (2019/2014)	Debt/GDP (2019)	Avg. Debt (2019)	Avg. Credit (2019)	Net Deficit (2019)
Austria	1%	2%	7,646	8,267	621
Belgium	0%	4%	21,857	7,408	-14450
Finland	-2%	3%	8,931	2,718	-6213
France	-5%	2%	64,244	103,250	39006
Germany	-1%	2%	72,480	69,782	-2697
Greece	-11%	1%	2,274	1,936	-338
Ireland	50%	5%	20,709	5,636	-15073
Italy	-7%	2%	43,374	34,018	-9356
Netherlands	2%	3%	29,868	40,511	10643
Portugal	4%	5%	12,277	3,291	-8986
Spain	2%	2%	27,935	66,493	38558
Sweden	-9%	1%	7,478	10,770	3292
Switzerland	-1%	3%	19,436	31,005	11570
U.K.	-8%	4%	102,157	55,580	-46577

Source: Author’s estimation based on World Development Indicators (WDI), Bank for International Settlements (BIS)

Table 2. Twenty-Four Countries' Network Model Output

Variable Name	Wave 1	Wave 2	Wave 3
gamma = 0.10; Theta = 0.80	Brazil		
gamma = 0.10; Theta = 0.85	Brazil, Turkey		
gamma = 0.10; Theta = 0.90	Brazil, Greece, Turkey		
gamma = 0.10; Theta = 0.95	Brazil, Greece, Italy, Sweden, Turkey	France	Spain, U.K.
gamma = 0.50; Theta = 0.80	Brazil		
gamma = 0.50; Theta = 0.85	Brazil, Turkey		
gamma = 0.50; Theta = 0.90	Brazil, Turkey		
gamma = 0.50; Theta = 0.95	Brazil, Greece, Turkey		

Source: Network model output; Note: gamma stands for cross-holdings ratio; theta stands for threshold

Table 3. Long-Term Steady State Debt Exposure Share

No.	Name	Share (%)	No.	Name	Share (%)
1	France*	0.2145	1	France	0.1497
2	Germany*	0.1691	2	U.S.	0.1450
3	U.K.*	0.1493	3	U.K.	0.1311
4	Spain*	0.1192	4	Japan	0.1209
5	Italy*	0.1000	5	German	0.1030
6	Netherlands*	0.0965	6	Spain	0.0694
7	Switzerland*	0.0640	7	Netherlands	0.0578
8	Austria*	0.0204	8	Canada	0.0530
9	Belgium*	0.0190	9	Italy	0.0505
10	Sweden*	0.0166	10	Switzerland	0.0477
11	Ireland	0.0102	11	Australia	0.0170
12	Portugal*	0.0102	12	Austria	0.0100
13	Finland*	0.0068	13	Belgium	0.0098
14	Greece*	0.0041	14	Sweden	0.0096
			15	Ireland	0.0057
			16	Portugal	0.0050
			17	Korea	0.0044
			18	Finland	0.0036
			19	Brazil*	0.0028
			20	Greece*	0.0020
			21	Turkey*	0.0008
			22	Chile	0.0005
			23	Panama	0.0005
			24	Mexico	0.0002

Source: Author's Markov process model output

Note: Network model Markov process output from scenario 2 and scenario 3. * stands for countries that ultimately fail

Table 4. Debt Cost between Benchmark and Shortest Path Scenario (Mil. US$)

From/To	Scenario	A	B	C	D	Shortest Path
A	Benchmark	0	1097	894	13969	
	Shortest Path	0	1097	894	12818	A-->C (894) -->D (11924)
B	Benchmark	2633	0	1355	241081	
	Shortest Path	2633	0	1355	13279	B-->C (1355) -->D (11924)
C	Benchmark	1494	798	0	11924	
	Shortest Path	1494	798	0	11924	
D	Benchmark	13929	18755	10091	0	
	Shortest Path	11585	10889	10091	0	D-->C (10091) -->A (1494)
	Shortest Path					D-->C (10091) -->B (798)

Source: Author's shortest path model output

Appendix A:

Deriving Values of the Initial Position, Mean, and Variances from Master Equation

Following Lefebvre (2007) and Gardiner (2009), we can derive the initial value of the master equation:

(Equation A1)

$$P_0(t) = \begin{cases} \left(\dfrac{\varphi^- - \varphi^- e^{(\varphi^- - \varphi^+)}}{\varphi^+ - \varphi^- e^{(\varphi^- - \varphi^+)}}\right)^N, & \forall \;\; \varphi^+ \neq \varphi^- \\ \left(\dfrac{\varphi^+ t}{1 + \varphi^+ t}\right)^N, & \forall \;\; \varphi^+ = \varphi^- \end{cases}$$

Taking the limit as time approaches infinity, we obtain

(Equation A2)

$$\lim_{t \to \infty} P_0(t) = \begin{cases} 1, & \forall \;\; \varphi^+ \leq \varphi^- \\ \left(\dfrac{\varphi^-}{\varphi^+}\right)^N, & \forall \;\; \varphi^+ > \varphi^- \end{cases}$$

The mean and variance of the loan addition and removal process are estimated as follows:
(Equation A3)

$$< n(t) > = \begin{cases} N e^{(\varphi^+ - \varphi^-)t}, & \forall \;\; \varphi^+ \neq \varphi^- \\ m, & \forall \;\; \varphi^+ = \varphi^- \end{cases}$$

Where <·> represents mean value.
(Equation A4)

$$\delta^2[n(t)] \; = \begin{cases} N\left[\dfrac{\varphi^+ + \varphi^-}{\varphi^+ - \varphi^-}\right] e^{(\varphi^+ - \varphi^-)t}\left(e^{(\varphi^+ - \varphi^-)t} - 1\right), & \forall \;\; \varphi^+ \neq \varphi^- \\ 2N\varphi^+ t, & \forall \;\; \varphi^+ = \varphi^- \end{cases}$$

Appendix B

Endogenous Default Rate

Following Anand et al. approach, the endogenous default rate can be derived as
$\mu = \frac{\xi}{2}\,\mathrm{erfc}(\eta)$ (Equation B1)
Where erfc(·) is the complimentary error function
$\eta = \frac{\Phi - 1 + q^0(\mu + \varphi^-)}{\sqrt{2(1+\Phi^2)(\mu + \varphi^-)}}$ (Equation B2)

Equation B2 shows that the impacts of external shocks on the balance sheet and network system are non-linear and non-monotonic. The lower boundary of network density can be represented by ε, $\varepsilon = \frac{1}{\varphi^-}$.

In the dynamic plane (b, l), with dl as the minimum payment obligation of a bank, the endogenous rate of bank failure can be defined as
$\mu = \xi \mathbb{p}(b^t - dl^t \leq -q^0)$ (Equation B3)

This equation explains the threshold moment of the endogenous default rate when a bank experiences a default. Since we have known a bank's liquid q^0, this leads us to figure out the probability of when η may be less than the liquid asset q^0, where $\eta = (b^t - dl^t)$. Denote $\Phi = \frac{b+q^0}{l}$. Assume η is a Gaussian distributed random variable with mean (1- Φ)/($\mu + \varphi^-$) and variance (1+$\Phi^2)/(\mu + \varphi^-$), and substitute these two values into the Gaussian distribution function, and we can derive

$\mu = \xi \frac{\sqrt{\mu+\varphi^-}}{\sqrt{2\pi(\Phi^2+1)}} \int_{q^0}^{\infty} \exp\left(-\frac{1}{2}\left(\eta - \frac{1-\Phi}{\mu+\varphi^-}\right)^2 \frac{\mu+\varphi^-}{\Phi^2+1}\right) d\eta = \frac{\xi}{2}$ erfc((η) (Equation B4)

Appendix C

Proof: In steady state, adding new loans and removing loans are constant at each state, that is,
$\varphi_i^+ = \varphi_{i+1}^+$, $\varphi_i^- = \varphi_{i+1}^-$. (Equation C1)
Where $0 \leq \varphi^+ \leq 1$, $0 < \varphi^- \leq 1$,
according to balance requirement in Markov Chain, the total inflow of new loans equals the total outflow of loans:
$\varphi^+\pi_i = \varphi^-\pi_{i+1}$ (Equation C2)
Where π_i represents probability distribution at state *i*.
We can further derive Equation C2, at state zero, we get
$\pi_1 = \frac{\varphi^+}{\varphi^-}\pi_0$ (Equation C3)
At state 1, the right-hand side represents the inflow of new loans and the left-hand side represents the outflow of loans.
And denote $\rho = \frac{\varphi^+}{\varphi^-}$ $\quad 0 \leq \rho < 1$ (Equation C4)
$\varphi^+\pi_0 + \varphi^-\pi_2 = \varphi^+\pi_1 + \varphi^-\pi_1$ (Equation C5)
Substitute Equation C1 and C2 into Equation C3 and rearrange, we can derive
$\pi_2 = \frac{\varphi^+}{\varphi^-}\pi_1 = \left(\frac{\varphi^+}{\varphi^-}\right)^2 \pi_0 = \rho p_1 = \rho^2 p_0$ (Equation C6)

By induction, we can obtain:
$\pi_n = \left(\frac{\varphi_{n-1}^+}{\varphi_n^-}\right)\pi_{n-1} \quad \forall\, n = 0,1,2 \ldots$ m (Equation C7a)
$\pi_n = \pi_0\rho^n$ (Equation C7b)

In order to find p_0, we can use the sum of all the probability states, which should be equal to 1:

$\sum_{i=1}^{\infty} \pi_i = 1$ (Equation C8)

Thus we have

$$\sum_{i=0}^{m} \pi_i = \pi_0 \left\{ \left(\frac{\varphi^+}{\varphi^-}\right)^0 + \left(\frac{\varphi^+}{\varphi^-}\right)^1 + \left(\frac{\varphi^+}{\varphi^-}\right)^2 + \left(\frac{\varphi^+}{\varphi^-}\right)^3 + \cdots + \left(\frac{\varphi^+}{\varphi^-}\right)^m \right\} = 1 \quad \text{(Equation C9)}$$

Substitute Equation C4 into Equation C9,

$\sum_{i=0}^{m} \pi_i = \pi_0(1 + \rho + \rho^2 + \rho^3 + \cdots + \rho^m) = 1$ (Equation C10)

$\pi_0 = \frac{1}{1+\rho^1+\rho^2+\rho^3+\cdots+\rho^m}$ (Equation C11)

Special case, if $\varphi^+ = \varphi^-$, or $\rho = 1$, then we can get $\pi_0 = \pi_i$, Equation C11 becomes

$\pi_0 = \frac{1}{1+m}$ (Equation C12)

This is a random walk or steady state process. It means the rate of adding new loans and removing loans has no change in the network system.

If $\varphi^+ < \varphi^-$, or $\rho < 1$,

$\lim_{m\to\infty} \sum_{n=0}^{m} \pi_n = \pi_0\{1 + \sum_{n=1}^{\infty} \rho^n\} = \frac{1}{1-\rho}\pi_0 = 1$ (Equation C13)

Then we can get

$\pi_0 = 1 - \rho$ (Equation C14)

Substitute Equation C13 into Equation C7b, we get

$\pi_n = \pi_0\rho^n = \rho^n(1-\rho)$ (Equation C15)

Equation C14 shows that when $\rho < 1$, that is $\frac{\varphi^+}{\varphi^-} < 1$, the rate of removing the number of loans is faster than the rate of adding the number of loans, and the network has a limiting state that ensures a finite number of new loans are added in the network system. In the long-run steady state, the expected value of this process is $E(X_n) = \frac{\rho}{1-\rho}$. When ρ is closer to 1, a very large number of new loans are added in the banking network system.

Appendix D. Proof of Proposition 2.

In the dynamic plane (l, b), we can derive that the mean of x satisfying

$$d_t < x(t) > = \partial_t \sum_{x=0}^{\infty} xP(x,t|x',t')$$

$$= \sum_{x=0}^{\infty} x[\pi_1(x-1)P(x-1,t|x',t') - \pi_2 P(x,t|x',t')]$$

$$+ \sum_{x=0}^{\infty} x[\pi_2(x+1)P(x+1,t|x',t') - \pi_1 P(x,t|x',t')]$$

$$= \sum_{x=0}^{\infty} [(x+1)\pi_1(x) + (x-1)\pi_2(x) - x(\pi_1(x) + \pi_1(x)]P(x,t|x',t')$$

We can obtain the mean value at the steady state

$$\frac{d}{dt} < x(t) \geq < \pi_1\big(x(t)\big) > - < (\pi_2\big(x(t)\big) >$$

Where <·> stands for mean value. If we only consider the deterministic component and neglect the fluctuation component,

$$\frac{dx}{dt} = \pi_1(x) - \pi_2(x)$$

Which means that at the long-term steady state, the rate of issuing new loans or renewing a loan should equal the rate of removing a loan or not renewing a contract whose maturity is reached.

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